

## GEOMETRY – SHEET 3 – Conics

1. Find the equations in Cartesian co-ordinates of:

- (i) the ellipse with foci at  $(\pm 2, 0)$  which passes through  $(0, 1)$ ;
- (ii) the hyperbola with asymptotes  $y = \pm 2x$  and directrices  $x = \pm 1$ ;
- (iii) the ellipse consisting of all points  $P$  such that  $|AP| + |BP| = 10$ , where  $A$  is  $(3, 0)$  and  $B$  is  $(-3, 0)$ ;
- (iv) the parabola with directrix  $x + y = 1$  and focus  $(-1, -1)$ .

[Hint: the point  $(x_0, y_0)$  is at a distance  $|ax_0 + by_0 + c|/\sqrt{a^2 + b^2}$  from the line  $ax + by + c = 0$ .]

2. Consider the parabola  $C$  with equation  $y^2 = 4ax$  and let point  $P = (at^2, 2at)$ . What is the gradient of  $C$  at  $P$  in terms of  $t$ ?

The focus  $F$  is at  $(a, 0)$ . Let  $l_1$  denote the line connecting  $P$  and  $F$ , and  $\theta_1$  denote the angle between  $l_1$  and the tangent to  $C$  at  $P$ . Show that

$$\cos \theta_1 = \frac{t}{\sqrt{t^2 + 1}}.$$

Let  $l_2$  denote the horizontal line through  $P$  and  $\theta_2$  denote the angle between  $l_2$  and the tangent to  $C$  at  $P$ . Show that  $\theta_1 = \theta_2$ . (Consequently any light beam emitted from the focus  $F$  will reflect horizontally after contact with a parabolic mirror.)

3. (i) The conic  $C$  is formed by intersecting the double cone  $x^2 + y^2 = z^2$  with the plane  $x + y + z = 1$ . Show that the point with position vector

$$\mathbf{r}(t) = (1 + (\sec t - \tan t)/\sqrt{2}, 1 + (\sec t + \tan t)/\sqrt{2}, -1 - \sqrt{2} \sec t)$$

lies on  $C$ .

(ii) Show that the vectors  $\mathbf{e}_1 = (1/\sqrt{6}, 1/\sqrt{6}, -\sqrt{2/3})$  and  $\mathbf{e}_2 = (-1/\sqrt{2}, 1/\sqrt{2}, 0)$  are of unit length, are perpendicular to one another and are parallel to the plane  $x + y + z = 1$ . Show further that

$$\mathbf{r}(t) = (1, 1, -1) + (a \sec t)\mathbf{e}_1 + (b \tan t)\mathbf{e}_2$$

where  $a$  and  $b$  are positive numbers to be determined.

(iii) Show that  $C$  has eccentricity  $2/\sqrt{3}$ , has foci  $(1, 1, -1) \pm 2\mathbf{e}_1$  and that the directrices, in parametric form, are  $(1, 1, -1) \pm 3\mathbf{e}_1/2 + \lambda\mathbf{e}_2$ .

4. By rotating the  $xy$ -axes appropriately, and subsequently completing the squares, show that the curve

$$6x^2 + 4xy + 9y^2 - 12x - 4y - 4 = 0,$$

is an ellipse and find its area.

5. Let  $\mathbf{v}$  and  $\mathbf{w}$  be independent vectors in  $\mathbb{R}^2$ . Show that

$$\mathbf{r}(t) = \mathbf{v} \cos t + \mathbf{w} \sin t,$$

where  $0 \leq t < 2\pi$ , is a parametrization of an ellipse.

6. (Optional) Let  $P$  be a point on an ellipse with foci  $F_1$  and  $F_2$ . Show that the two lines  $F_1P$  and  $F_2P$  make the same angle with the normal to the ellipse.

This means that if you are playing billiards on an elliptical table, with a pocket on one focus and the ball placed on the other focus, every shot will go in! (Assuming it is hit sufficiently hard.) Click here for a cool YouTube video showing this.

Consider: is there a related statement for a hyperbola?